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Topological Properties of Non-Periodic Kronig-Penney Model

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Outline

1. ***Introduction***

2. ***Review of Topology in Condensed Matter Physics***

3. ***Main part : My Research Topic***

1. Introduction

1.1. Self Introduction

1.2. Why I came here?

1.3. My previous study (Not research)

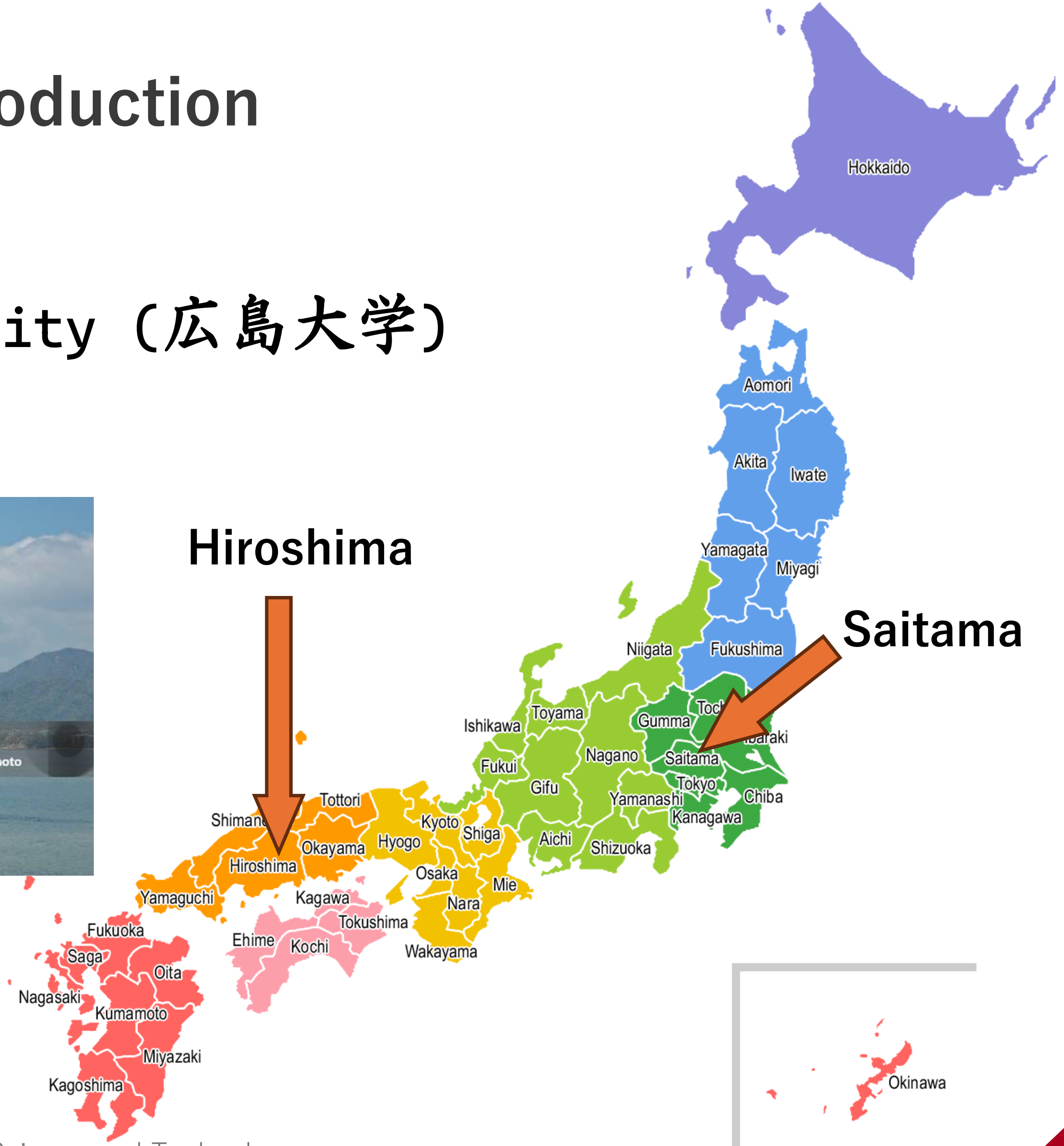
- *1.3.1. Topological Insulator*
- *1.3.2. Bose-Hubbard Model*
- *1.3.4. Kosterlitz-Thouless transition*

Self Introduction

Name: Naoki Itsui
Age: 21 years old
University: Hiroshima University (広島大学)
Hometown: Saitama (埼玉)

Hiroshima:

Saitama:



Self Introduction

- ✓ Research Unit in Hiroshima University → **Condensed matter Theory Group**
- ✓ Our Unit member are especially interested in **Topological Phenomena**

PI (supervisor): Prof. Yasuhiro Tada

- ➢ Superconductivity
- ➢ Quantum criticality
- ➢ Dirac electrons
- ➢ Heavy fermions
- ➢ Topological systems
- ➢ Quantum magnets
- ➢ Spin liquids

M2, 1 person: Quantum Spin liquids
M1, 1 person: Fractional quantum Hall effect
B4, 4 people: Undecided

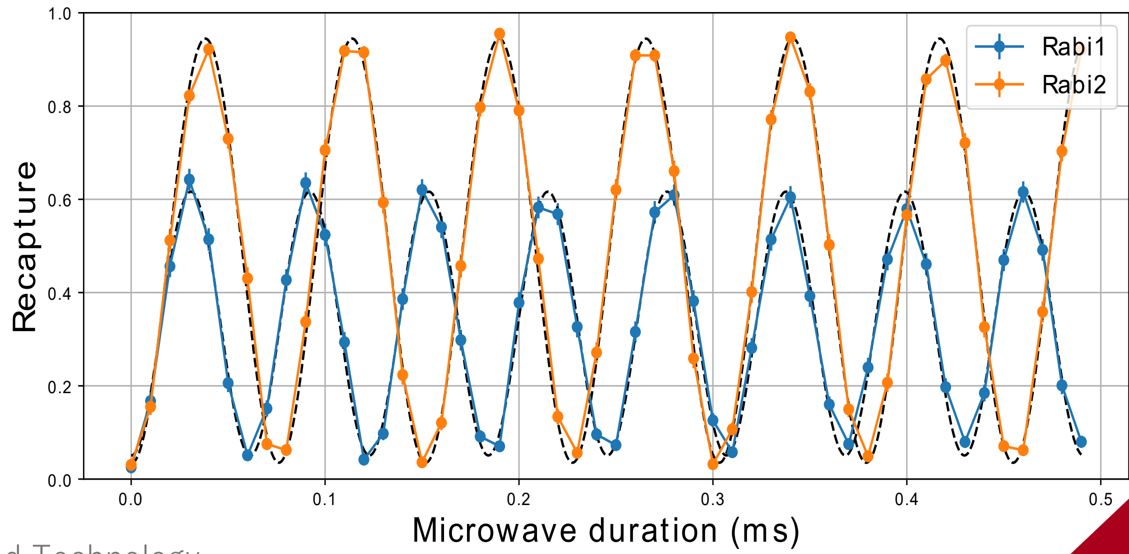
※
M2 = 2^{nd} year master student
M1 = 1^{st} year master student
B4 = 4^{th} year bachelor student
In Japan Graduate school
= two years mater degrees + three years Ph.D. degrees

Why I came here?

Last year, I took a lecture titled
An Introduction to Ultracold Atom Physics for Condensed Matter Physics and Quantum Computation
by Prof. Tomita (Institute for Molecular Science Ohmori group)

→ I started to have interested in Ultracold Atom Physics
+
I knew the OIST Research Internship program and found the QSU group

I also, I went to Institute for Molecular Science Ohmori group for three days and did a easy experiment (observation of Rabi oscillation) based on Prof. Tomita's instruction.



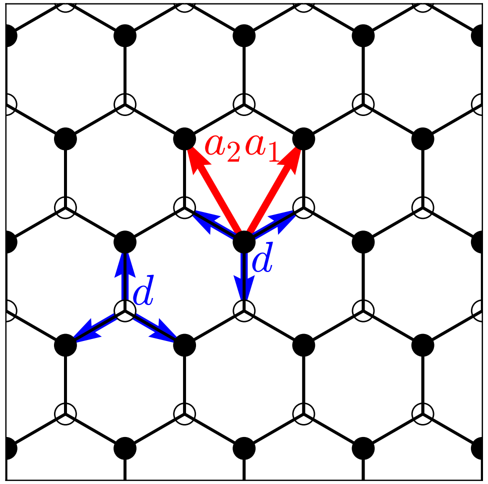
My previous study : Topological Insulator

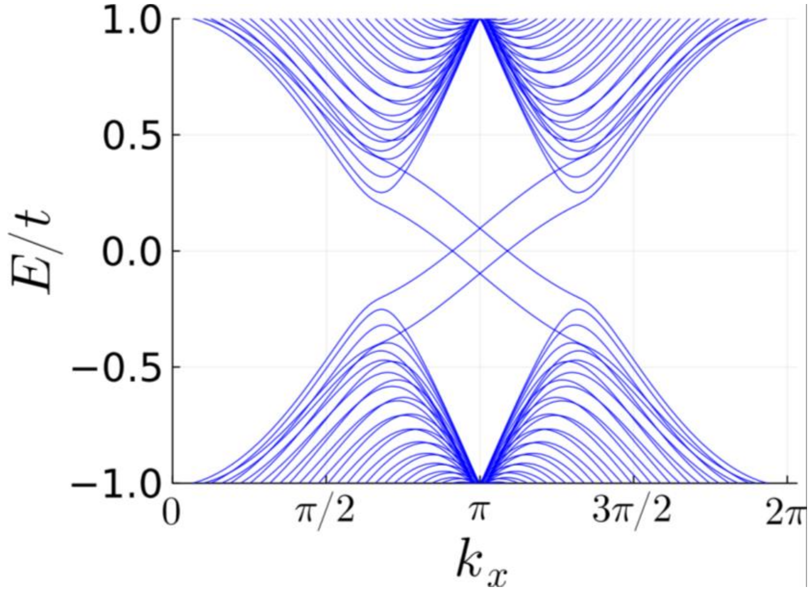
Kane-Mele model is the first theoretical proposed Topological Insulator (generalized Quantum Spin Hall Insulator)

$$H_{\mathrm{KM}} = t \sum_{\langle i,j \rangle} c_i^\dagger c_j + i\lambda_{\mathrm{SO}} \sum_{\langle\langle i,j \rangle\rangle} \nu_{ij} c_i^\dagger s^z c_j + \lambda_v \sum_i \xi_i c_i^\dagger c_i$$

$$(-1)^\nu = \prod_{i=1}^{4} \zeta(\boldsymbol{\Lambda_i})$$

$\zeta(\boldsymbol{\Lambda_i})$: Parity at time reversal momentum $\boldsymbol{\Lambda_i}$





My previous study : Bose-Hubbard Model

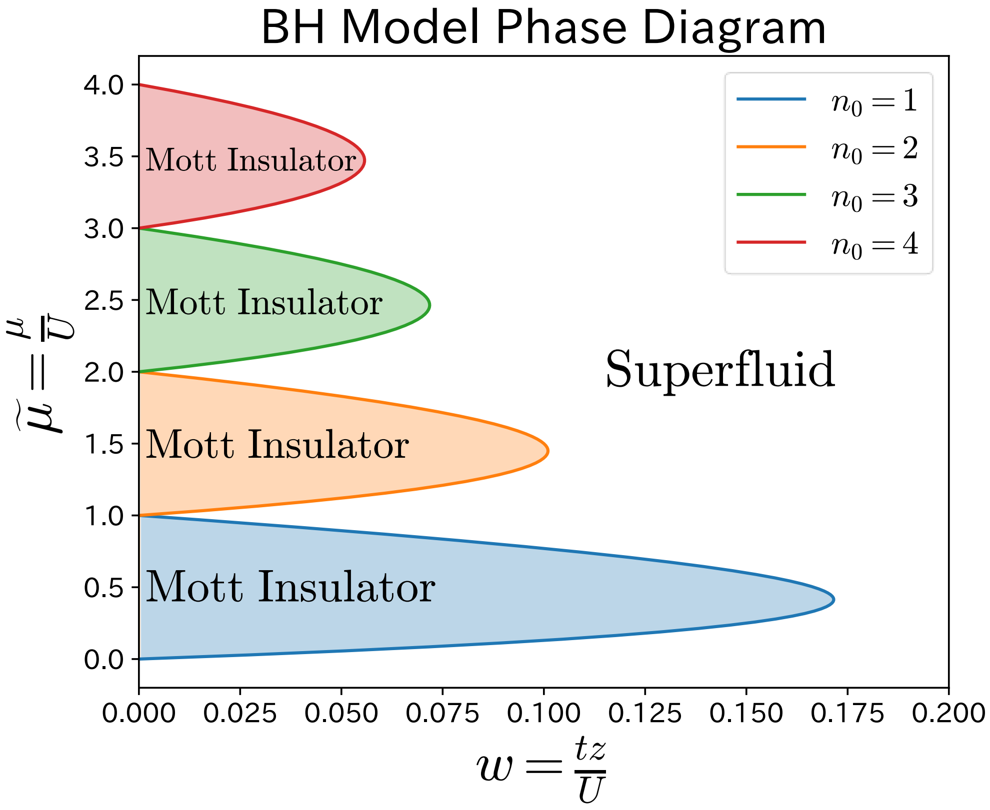
Calculation was so heavy I could not completely calculate fourth perturbation

$$H_{\mathrm{BH}} = \sum_{\boldsymbol{R}} (\epsilon_{\boldsymbol{R}} - \mu) a^\dagger_{\boldsymbol{R}} a_{\boldsymbol{R}} - t \sum_{\boldsymbol{R}} \sum_{i=1}^{D} (a^\dagger_{\boldsymbol{R}} a_{\boldsymbol{R}+\boldsymbol{e}_i} + \mathrm{h.c.}) + \frac{U}{2} \sum_{\boldsymbol{R}} \hat{n}_{\boldsymbol{R}} (\hat{n}_{\boldsymbol{R}} - 1)$$

$$\Xi_{\mathrm{BH}} = \int \mathcal{D}\alpha^* \mathcal{D}\alpha \mathcal{D}\Psi^* \mathcal{D}\Psi \exp\left[-S_{\mathrm{BH}}[\alpha^*, \alpha, \Psi^*, \Psi]\right]$$

$$\begin{aligned} S_{\mathrm{BH}}[\alpha^*, \alpha, \Psi^*, \Psi] = &\int_0^\beta d\tau \Bigg\{ \sum_{\boldsymbol{R}} \left(\alpha^*_{\boldsymbol{R}} (\partial_\tau + \epsilon_{\boldsymbol{R}} - \mu) \alpha_{\boldsymbol{R}} + \frac{U}{2} \alpha^*_{\boldsymbol{R}} \alpha^*_{\boldsymbol{R}} \alpha_{\boldsymbol{R}} \alpha_{\boldsymbol{R}} \right) \\ &- \sum_{\boldsymbol{R}} (\alpha^*_{\boldsymbol{R}} \Psi_{\boldsymbol{R}} + \Psi^*_{\boldsymbol{R}} \alpha_{\boldsymbol{R}}) + \sum_{\boldsymbol{R},\boldsymbol{R}'} \Psi^*_{\boldsymbol{R}} t^{-1}_{\boldsymbol{R},\boldsymbol{R}'} \Psi_{\boldsymbol{R}'} \Bigg\} \\ = &S_0 + S_\Psi \end{aligned}$$

$$\begin{cases} S_0 = \displaystyle\int_0^\beta d\tau \sum_{\boldsymbol{R}} \left(\alpha^*_{\boldsymbol{R}} (\partial_\tau + \epsilon_{\boldsymbol{R}} - \mu) \alpha_{\boldsymbol{R}} + \frac{U}{2} \alpha^*_{\boldsymbol{R}} \alpha^*_{\boldsymbol{R}} \alpha_{\boldsymbol{R}} \alpha_{\boldsymbol{R}} \right) \\ S_\Psi = \displaystyle\int_0^\beta d\tau \left[- \sum_{\boldsymbol{R}} (\alpha^*_{\boldsymbol{R}} \Psi_{\boldsymbol{R}} + \Psi^*_{\boldsymbol{R}} \alpha_{\boldsymbol{R}}) + \sum_{\boldsymbol{R},\boldsymbol{R}'} \Psi^*_{\boldsymbol{R}} t^{-1}_{\boldsymbol{R},\boldsymbol{R}'} \Psi_{\boldsymbol{R}'} \right] \end{cases}$$

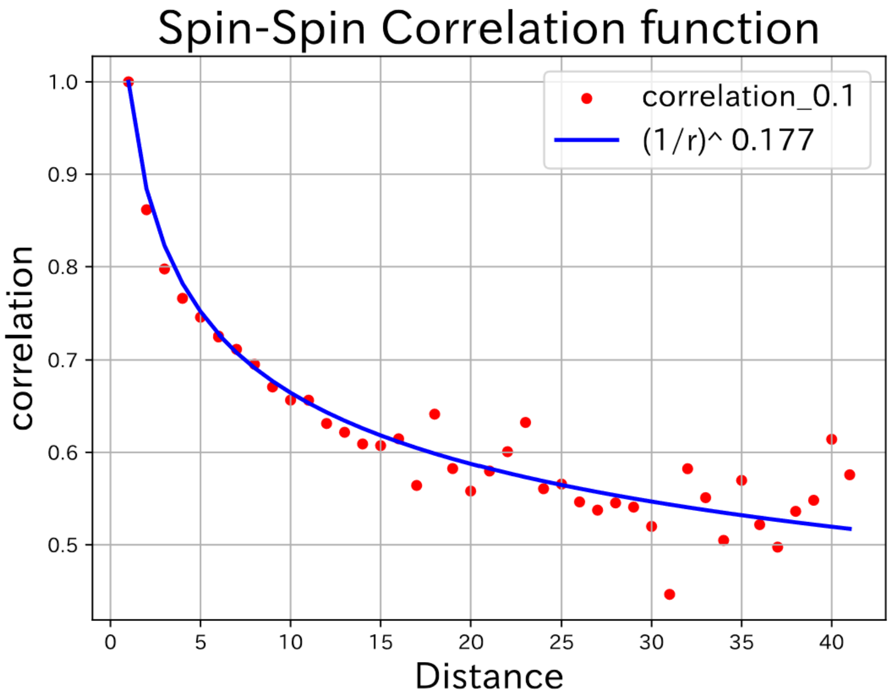


My previous study : Kosterlitz-Thouless transition

$$H_{\mathrm{KT}} = -J \sum_{\langle i,j \rangle} (S_i^x S_j^x + S_i^y S_j^y) = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$$T < T_c : \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle = \left(\frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\right)^{-\frac{1}{2\pi J \beta}}$$

$$T > T_c : \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle = \exp\left(-\frac{|\boldsymbol{r}_i - \boldsymbol{r}_j|}{\xi}\right), \quad \xi = \left(\log \frac{2}{\beta J}\right)^{-1}$$



Not good

2. Review of the Topology in Condensed Matter Physics

2.1. What is the topology?

2.2. Topology in Condensed Matter Physics

2.3. Review of Quantum Hall Effect and TKNN formula

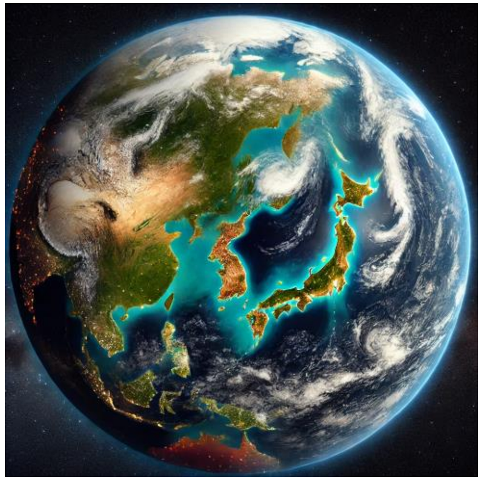
2.4. Review of the Topological pump

What is the Topology?

✓"Equivalent" in topology is if and only if two shapes can be **Continuously Deformed** to each other

✓Topological number in this case is a hole that does not change in continuous deformation

Continuously Deform



Topology in Condensed Matter Physics

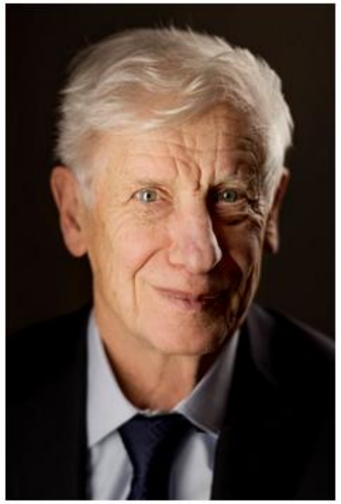
Topology in Condensed Matter Physics

Example : Nobel prize in physic in 2016

✓TKNN Formula

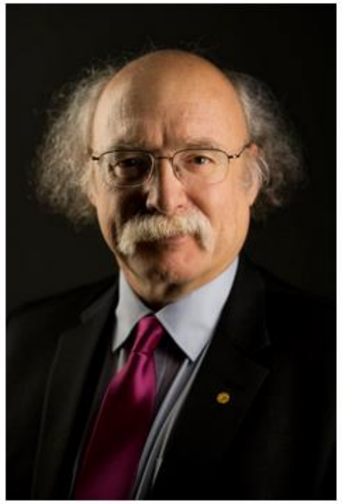
✓Haldane Conjecture

✓Kosterlitz-Thouless transition



David J. Thouless

Prize share: 1/2

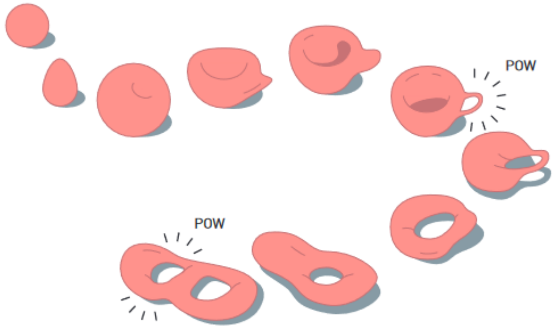


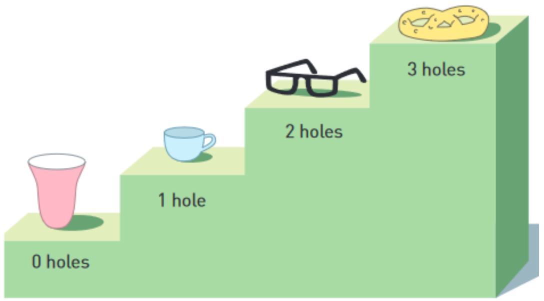
F. Duncan M. Haldane

Prize share: 1/4

J. Michael Kosterlitz

Prize share: 1/4





Topology in Condensed Matter Physics

Before starting main slides.....

There are a lot of interesting topics in Topological Condensed matter Physics

- Fractional Quantum Hall Effect
- Topological Insulator and Topological Superconductor
- Quantum spin liquid
- Classifications of Topological Phase of Matter (Topological order, SPT, SET...)

See a good review : X. G. Wen, Rev. Mod. Phys. **89**, 041004 (2017)

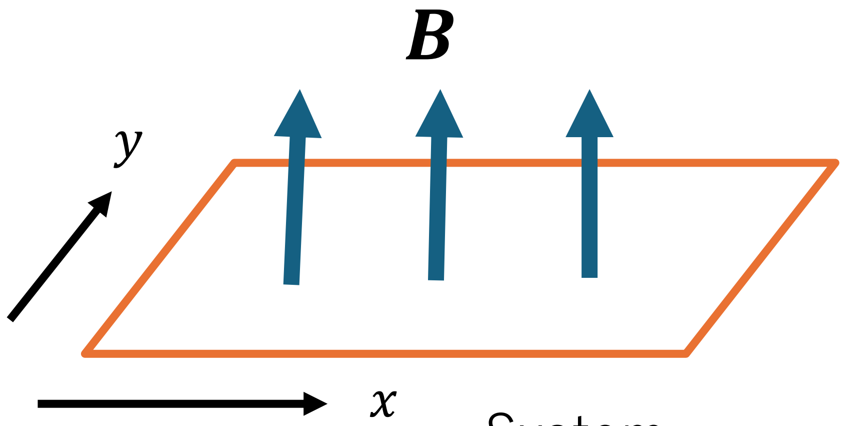
Review of Quantum Hall Effect and TKNN formula

2D system with vertical magnetic field.

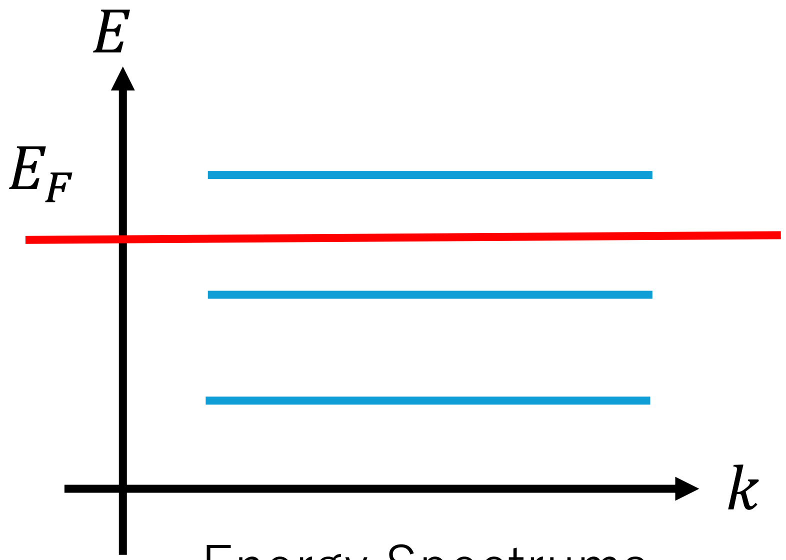
Energy eigenvalues are quantized (Landau Level)

Hall conductivity are quantized by integer C
(**Chern Number**)

$$\sigma_{xy} = \frac{e^2}{h} C \qquad (j_x = \sigma_{xy} E_y)$$



System



Energy Spectrums

Review of Quantum Hall Effect and TKNN formula

$$C = \sum_{n=1}^{\mathrm{occ}} C_n = \sum_{n=1}^{\mathrm{occ}} \frac{1}{2\pi} \int_{\mathrm{BZ}} d^2k \, \mathcal{B}_n(\boldsymbol{k})$$ **Chern Number**

$$\mathcal{B}_n(\boldsymbol{k}) = \nabla_{\boldsymbol{k}} \times \mathcal{A}_n(\boldsymbol{k})$$ Berry Curvature

$$[\mathcal{A}_n(\boldsymbol{k})]_i = i \langle u_{n,\boldsymbol{k}} | \frac{\partial}{\partial k_i} | u_{n,\boldsymbol{k}} \rangle$$ Berry Connection

The Chern number is Topological number.

This mathematical formula is called **TKNN formula**

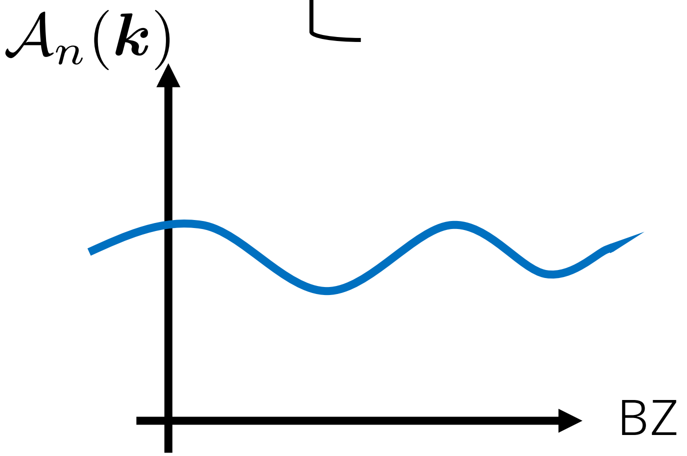
Review of Quantum Hall Effect and TKNN formula

Why Chern Number is topological number? (Physical meaning)

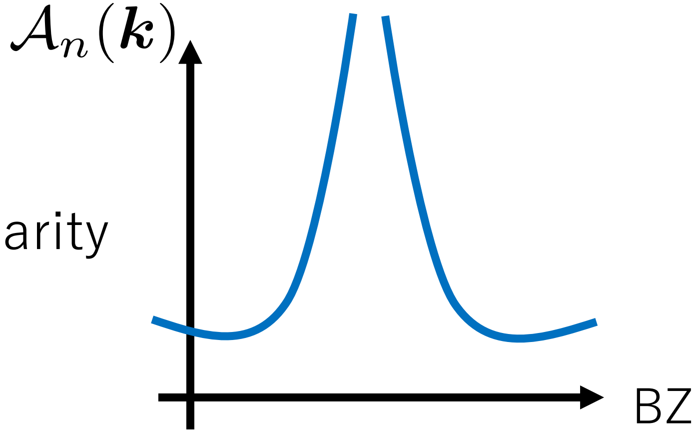
Chern Number corresponds to whether **Berry Connection** $\mathcal{A}_n(\boldsymbol{k})$ **can be smoothly defined or not by one gauge** in the BZ.

Chern number =
- Zero : can be smoothly defined by one gauge
- Non-zero: cannot be smoothly defined by one gauge

$C = 0$
smooth



$C \neq 0$
singularity



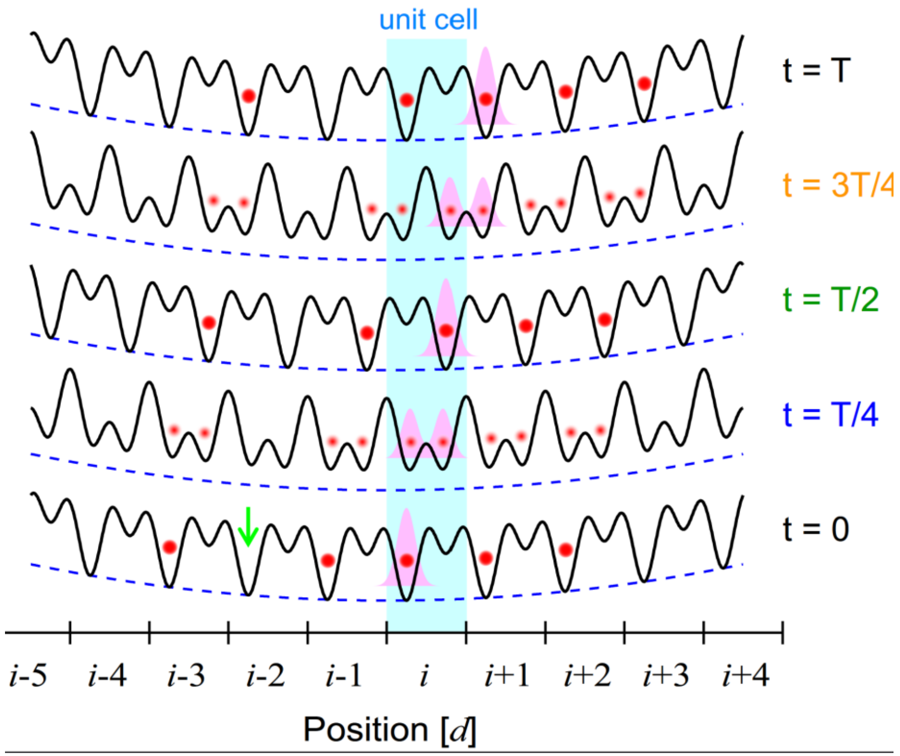
Review of the Topological pump

What is Topological Pump (or Thouless Pump)?

✓We consider 1+1D system and the potential is periodic both in time and space.

✓The charge pumped by the time dependent potential is a topological quantum number.

✓This topological quantum number is Chern number



Nakajima et al., Nature Phys **12**, 296–300 (2016)

Review of the Topological pump

We define the Wannier function by Fourier transformation of Bloch wave function.

$$|\boldsymbol{R}n\rangle = \frac{V}{(2\pi)^3}\int_{\mathrm{BZ}} d^3k\ e^{-i\boldsymbol{k}\cdot\boldsymbol{R}}\,|\psi_{n,\boldsymbol{k}}\rangle$$

$|\psi_{n,\boldsymbol{k}}\rangle$ is the Bloch wave function

Properties:

- ✓ Spans the same Hilbert space that spanned by Bloch wave function
  ➡ **Contains the same information!**
- ✓ Localized basis
  ➡ **can be used to construct the Maximally localized basis in real space**
- ✓ Depends on the gauge of Bloch function
  ➡ **No longer eigenstates of Hamiltonian**

Review of the Topological pump

$$P_\rho(t) = \frac{1}{L}\sum_{n=1}^{\text{occ}} \langle R=0n,t|\,\hat{x}\,|R=0n,t\rangle$$

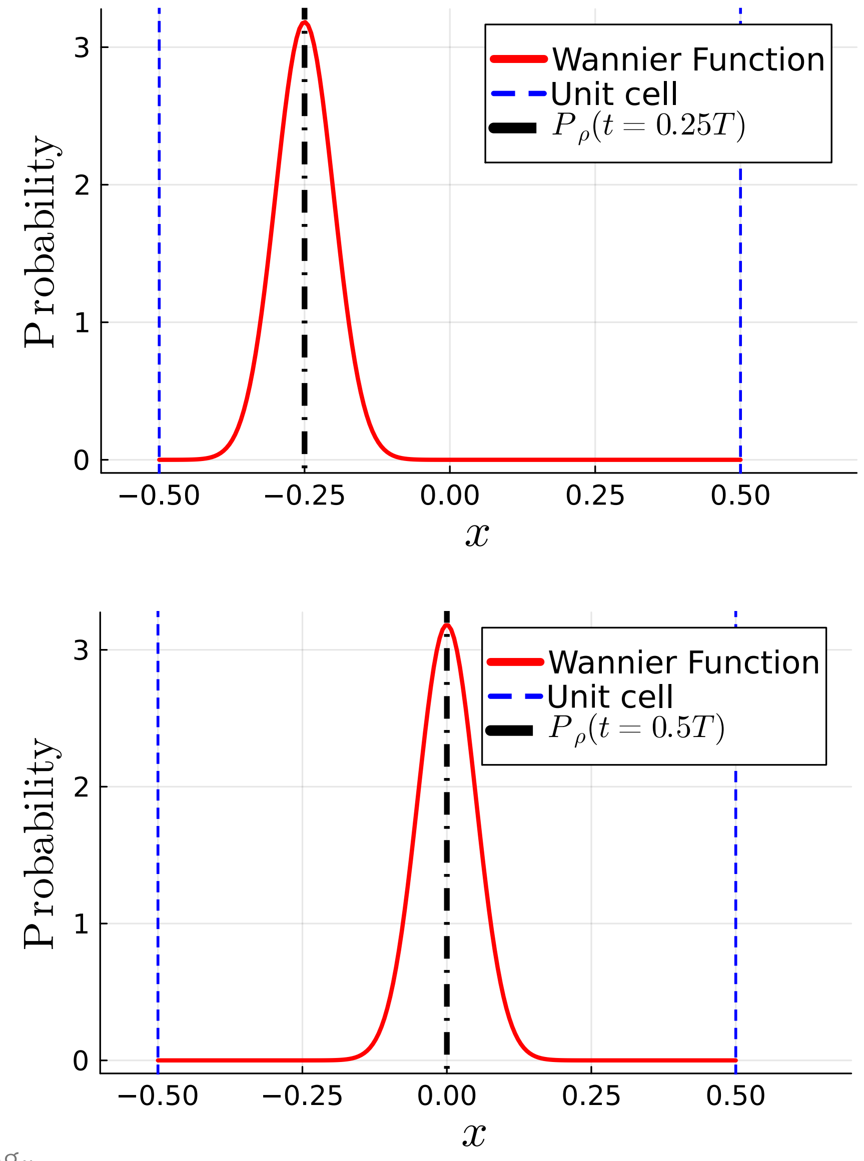
$|Rn,t\rangle$ is Wannier Functions at unit cell R

$$P_\rho(T) - P_\rho(0) = \frac{1}{2\pi}\sum_{n}^{\text{occ}}\int_0^T dt\int_{-\pi}^{\pi} dk\;\mathcal{B}_n(k,t)$$

$$\mathcal{B}_n(k,t) = \frac{\partial}{\partial k}[\mathcal{A}_n(k,t)]_t - \frac{\partial}{\partial t}[\mathcal{A}_n(k,t)]_k$$

$$[\mathcal{A}_n(k,t)]_k = i\,\langle u_{n,k}(t)|\,\frac{\partial}{\partial k}\,|u_{n,k}(t)\rangle$$

$$[\mathcal{A}_n(k,t)]_t = i\,\langle u_{n,k}(t)|\,\frac{\partial}{\partial t}\,|u_{n,k}(t)\rangle$$



Review of the Topological pump

Example : Rice-Mele Model

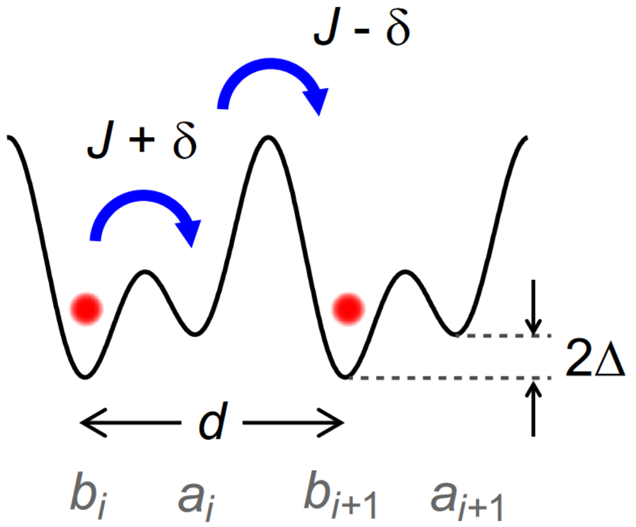
$$\hat{H}_{\mathrm{RM}} = \sum_i \left(-(J+\delta)\hat{a}_i^\dagger \hat{b}_i - (J-\delta)\hat{a}_i^\dagger \hat{b}_{i+1} + \mathrm{h.c.} + \Delta(\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i) \right)$$

After Fourier transformation, the Hamiltonian is

$$\hat{H}_{\mathrm{RM}} = \sum_k \begin{pmatrix} a_k^\dagger & b_k^\dagger \end{pmatrix} \mathcal{H}(k,t) \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$\mathcal{H}(k,t) = \boldsymbol{R}(k,t)\cdot\boldsymbol{\sigma},\ \ \boldsymbol{\sigma} \text{ is Pauli matrices}$$

$$\boldsymbol{R}(k,t) = \begin{pmatrix} -2J\cos\frac{kd}{2} \\ 2\delta(t)\sin\frac{kd}{2} \\ \Delta(t) \end{pmatrix}$$



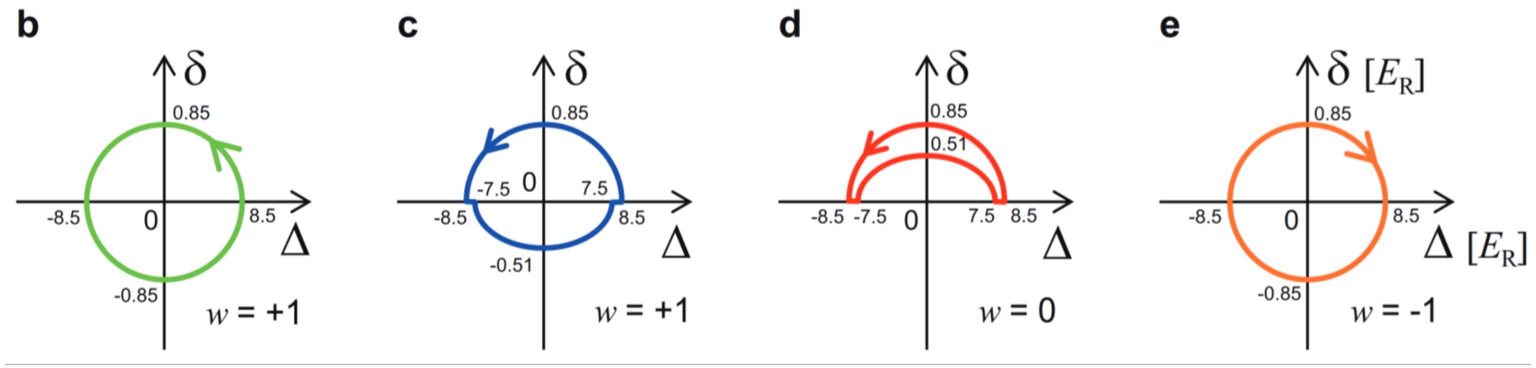
Nakajima et al., Nature Phys **12**, 296–300 (2016)

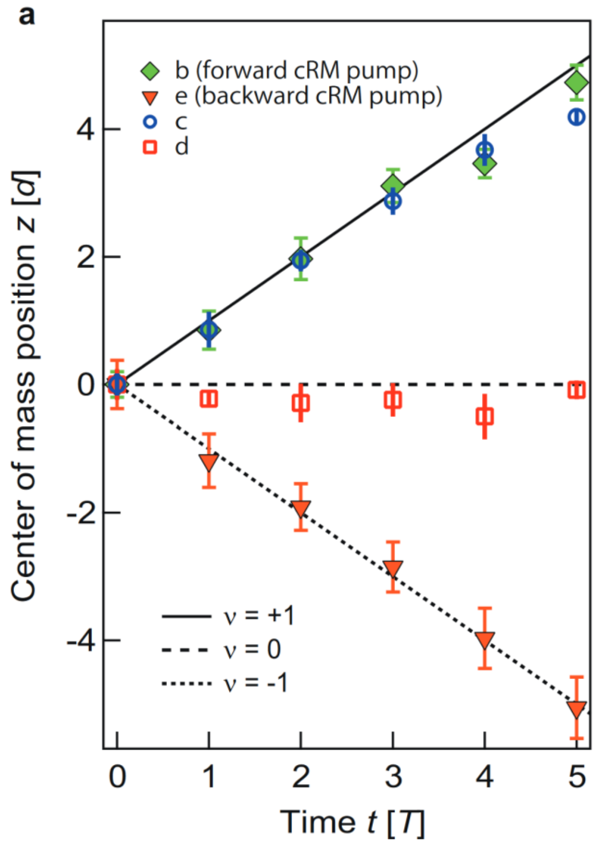
Review of the Topological pump

Example : Rice-Mele Model

After some calculation,

the total charge pumping = the winding number of (Δ, δ) space





Experimentally observed in

Nakajima et al., Nature Phys **12**, 296–300 (2016)

3. Main part : My Research Topic

3.1. Problem and Motivation

3.2. Local Chern Marker

3.3. Model

3.4. Results

3.4.1. $\boldsymbol{\beta} = \frac{3}{5}$ *: Periodic Case*

3.4.2. $\boldsymbol{\beta} = \frac{2}{\sqrt{5}+1}$ *: Non-periodic case*

3.4.3. $\boldsymbol{\beta} = \frac{1}{3}, \frac{2}{3}$ *: Domain Wall Case*

Problem and Motivation

- Chern number can only be defined in the quasi-momentum space.
- Chern number assumes that the periodicity of crystal.
- We want to calculate the Chern number without periodicity like quasi-crystal.

Local Chern Marker

We define the Local Chern marker.

$$\mathcal{M}_1(x,t) = \frac{1}{L}\mathrm{Tr}_x[\hat{P}(0)\hat{U}^\dagger(t)\hat{x}\hat{U}(t)\hat{P}(0)]$$

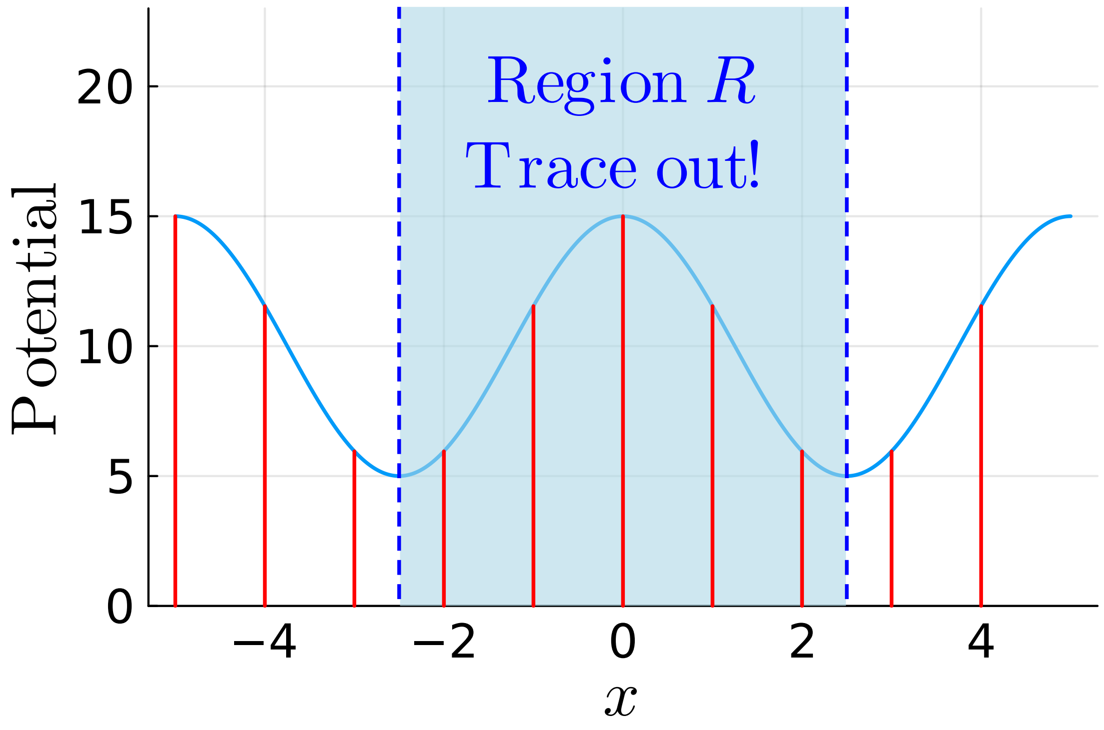
L is the length of the system

$$\mathrm{Tr}_x[\hat{\mathcal{O}}] = \sum_\alpha [\langle x| \otimes \langle \alpha|]\ \hat{\mathcal{O}}\ [|x\rangle \otimes |\alpha\rangle]$$

α is the internal degrees of freedom

$\hat{P(t)}$ is the projection operaotr

$\hat{U}(t)$ is the adiabatice time evolution operator



Local Chern Marker at $x = 0$

The region R is 5

Positions within R become internal degrees of freedom

Local Chern Marker

If **the system has translational invariant** (= **periodically**), the difference in one period of the local Chern marker corresponds to the difference in one period of the charge pumping.

$$P_\rho(T) - P_\rho(0) = \mathcal{M}_1(x,T) - \mathcal{M}_1(x,0)$$

$$\text{If the system has translational invariant}$$

Local Chern Maker is the generalization of Chern number in 1+1D!!!

Model

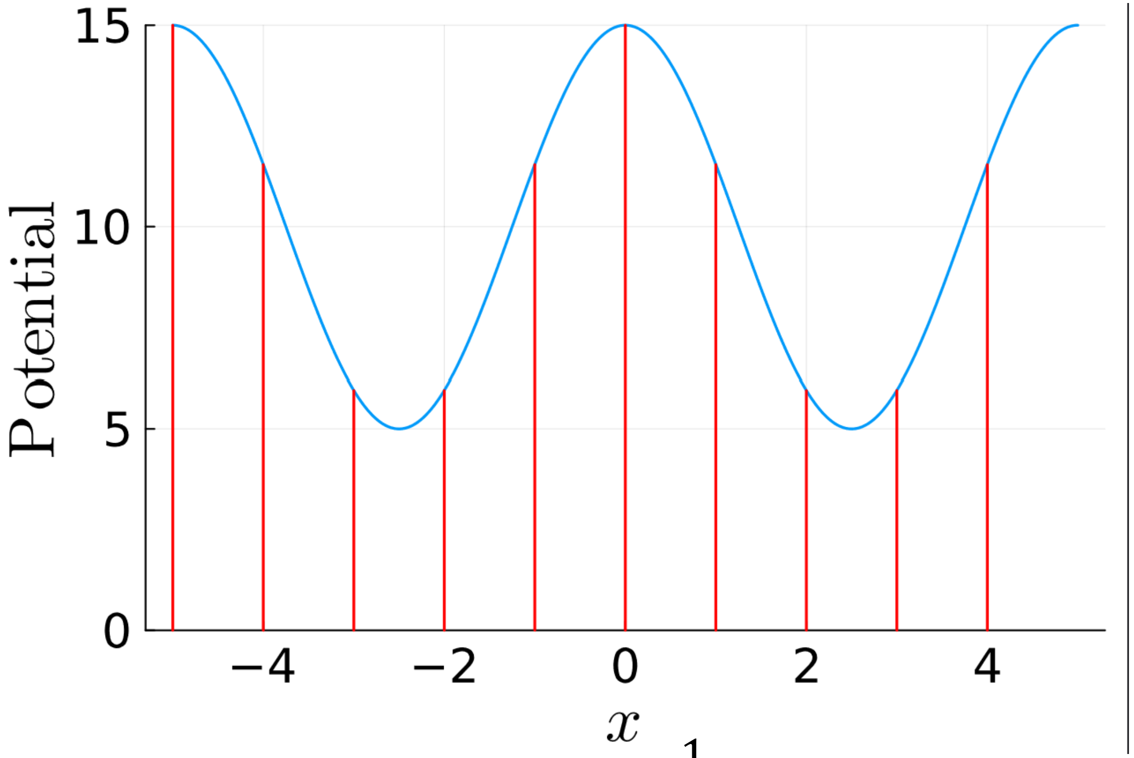
Consider Modulated Kronig-Penney Model with Open Boundary Condition.

$$H = -\frac{d^2}{dx^2} + \sum_{j \in \text{system}} h_j^{\alpha\beta\gamma\Delta} \delta(x - x_j^\Delta)$$

$$h_j^{\alpha\beta\gamma\Delta} = h_0(1 + \alpha\cos(2\pi\beta x_j^\Delta + \gamma))$$

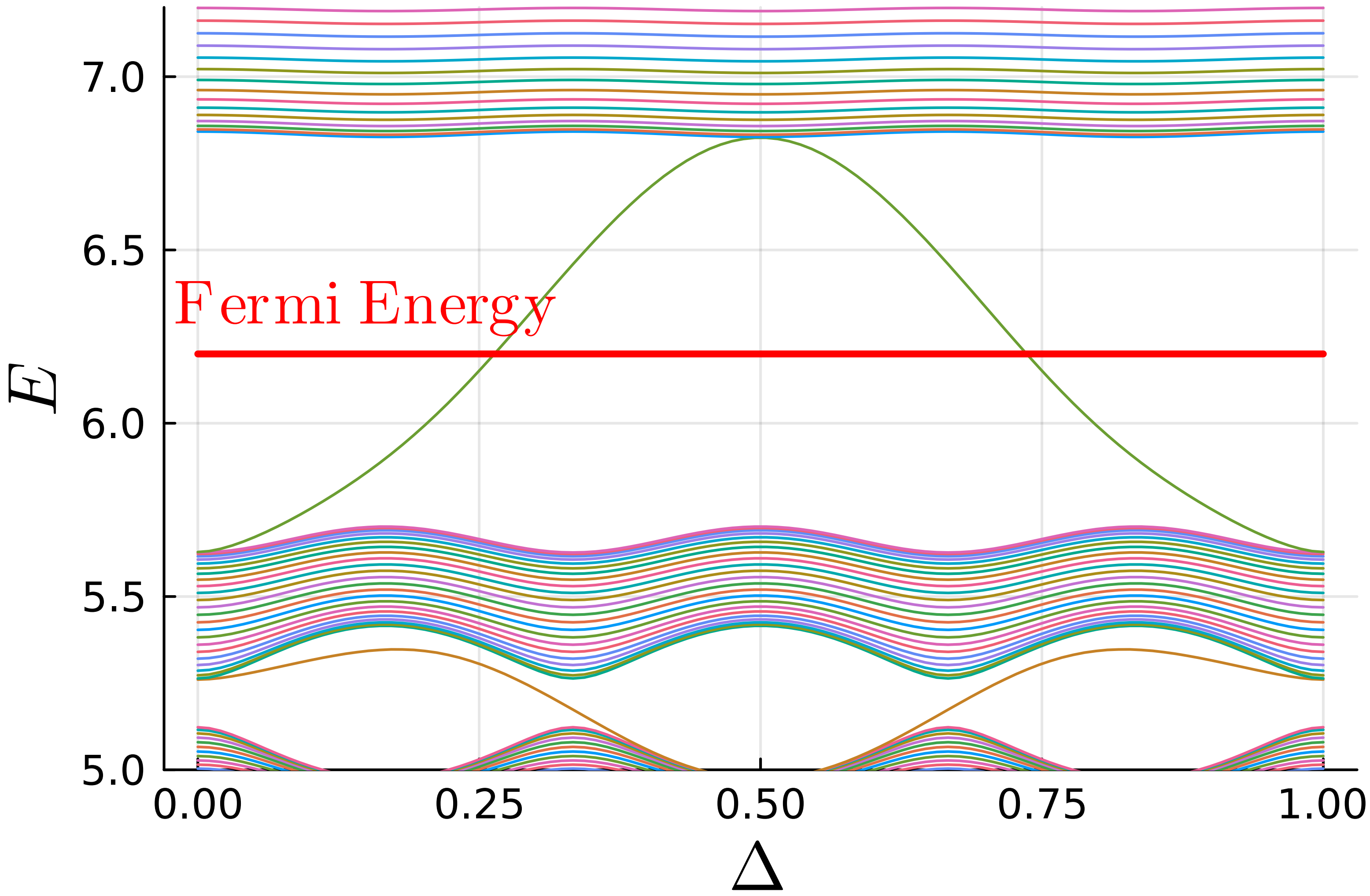
$$x_j^\Delta = x_j + \Delta$$

✓ 1+1D model
✓ $\Delta \to t$
✓ Period $T = 1$

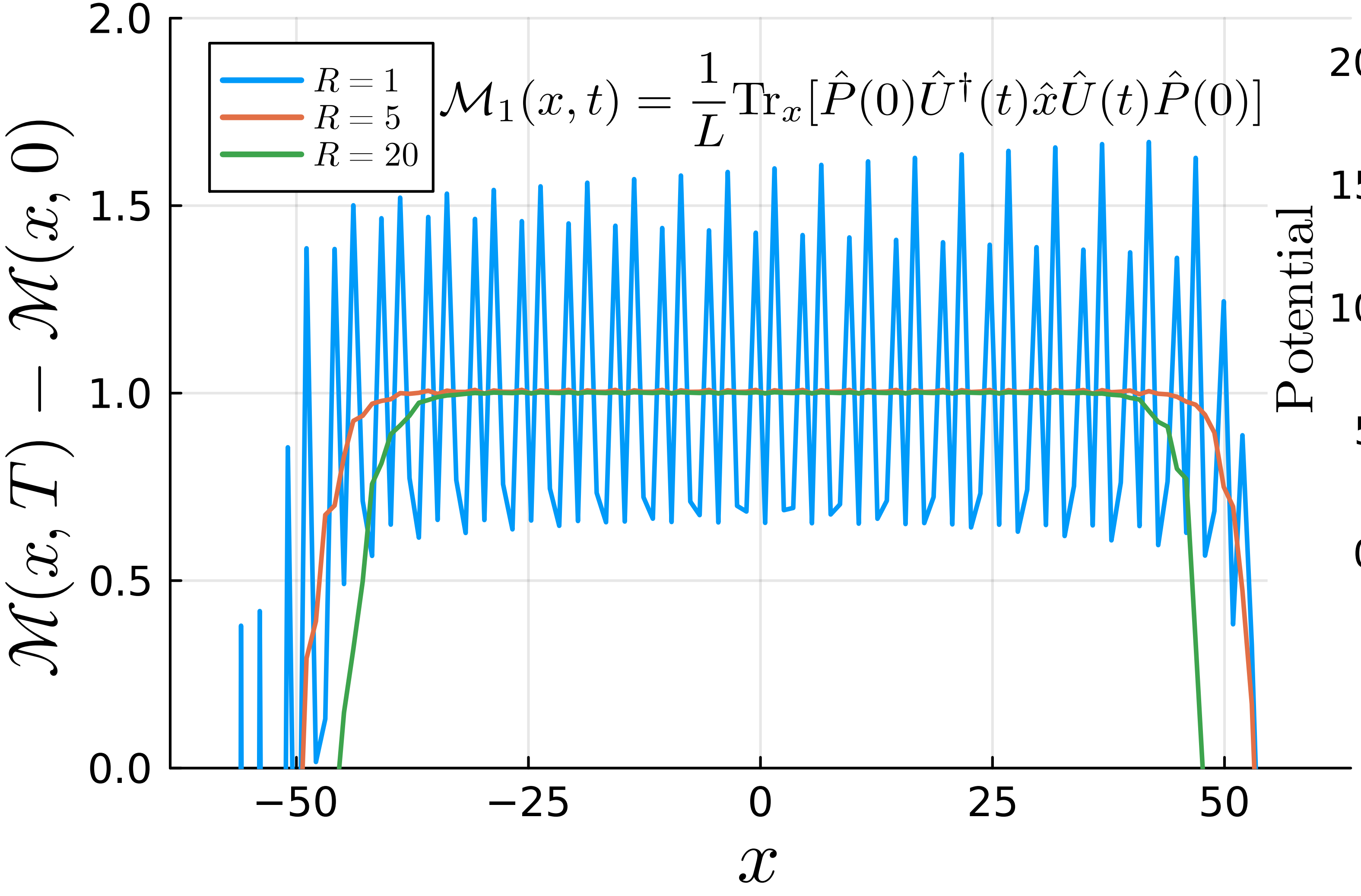


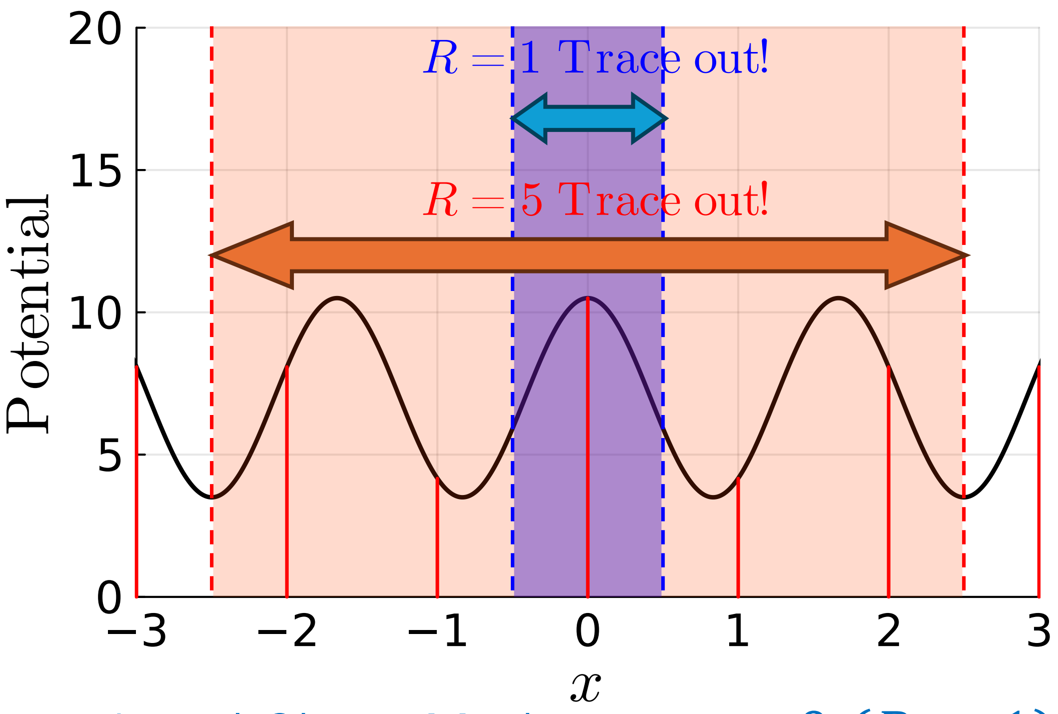
$h_0 = 10, \alpha = 0.5, \beta = \frac{1}{5}, \gamma = 0, \Delta = 0$

Results of $\beta = \frac{3}{5}$: Periodic Case



Results of $\beta = \frac{3}{5}$: Periodic Case

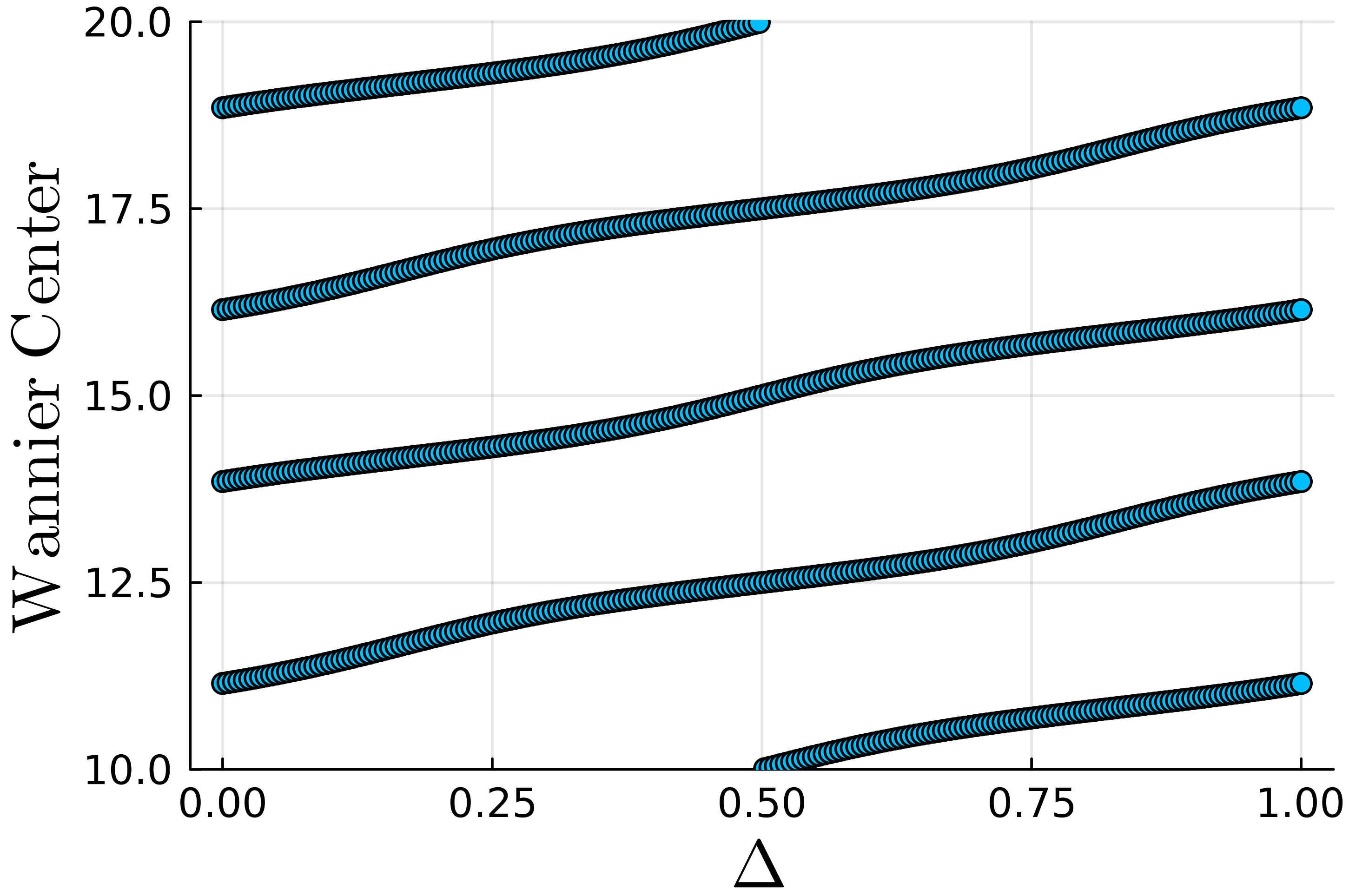




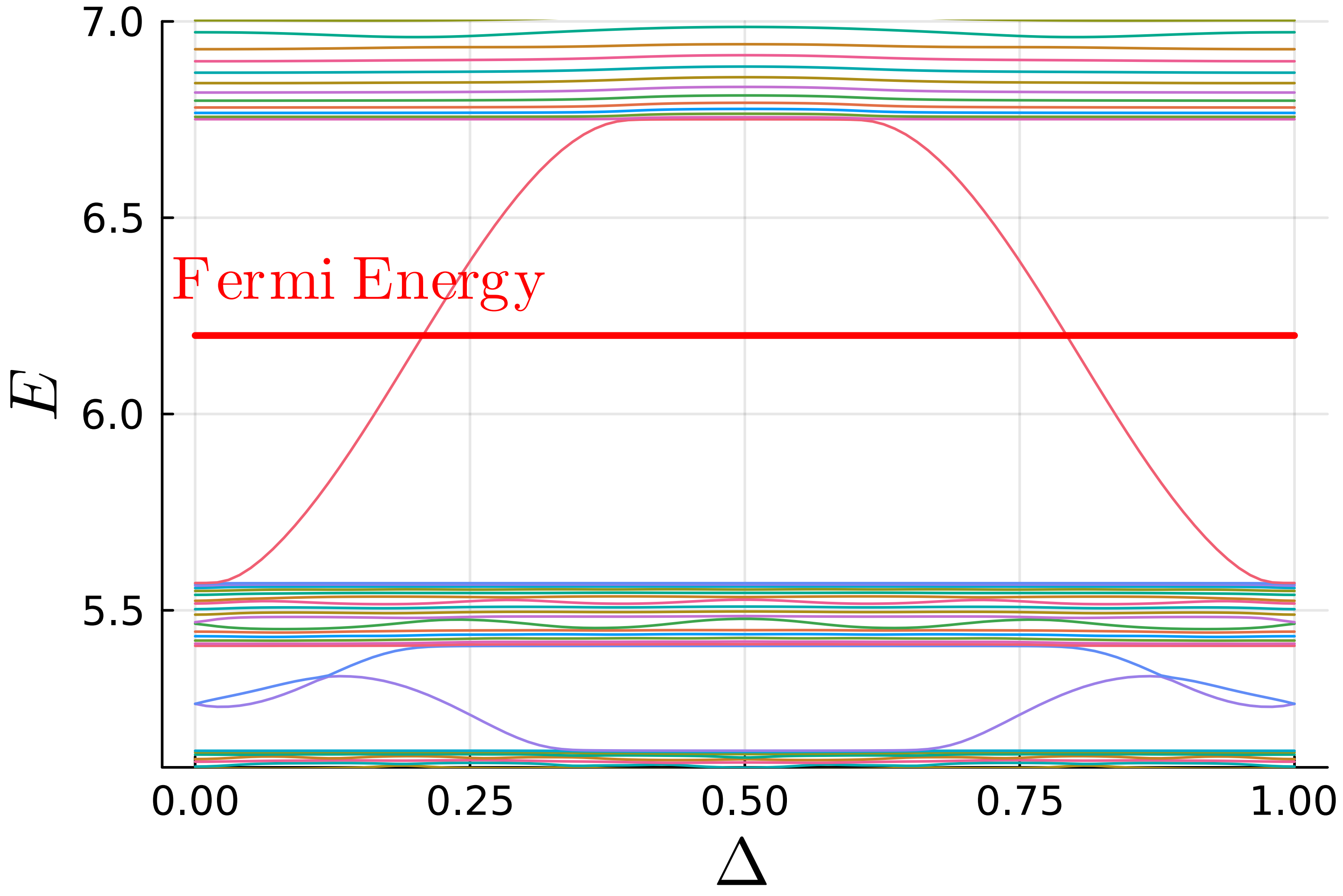
Local Chern Marker at $x = 0$ $(R = 1)$

Local Chern Marker at $x = 0$ $(R = 5)$

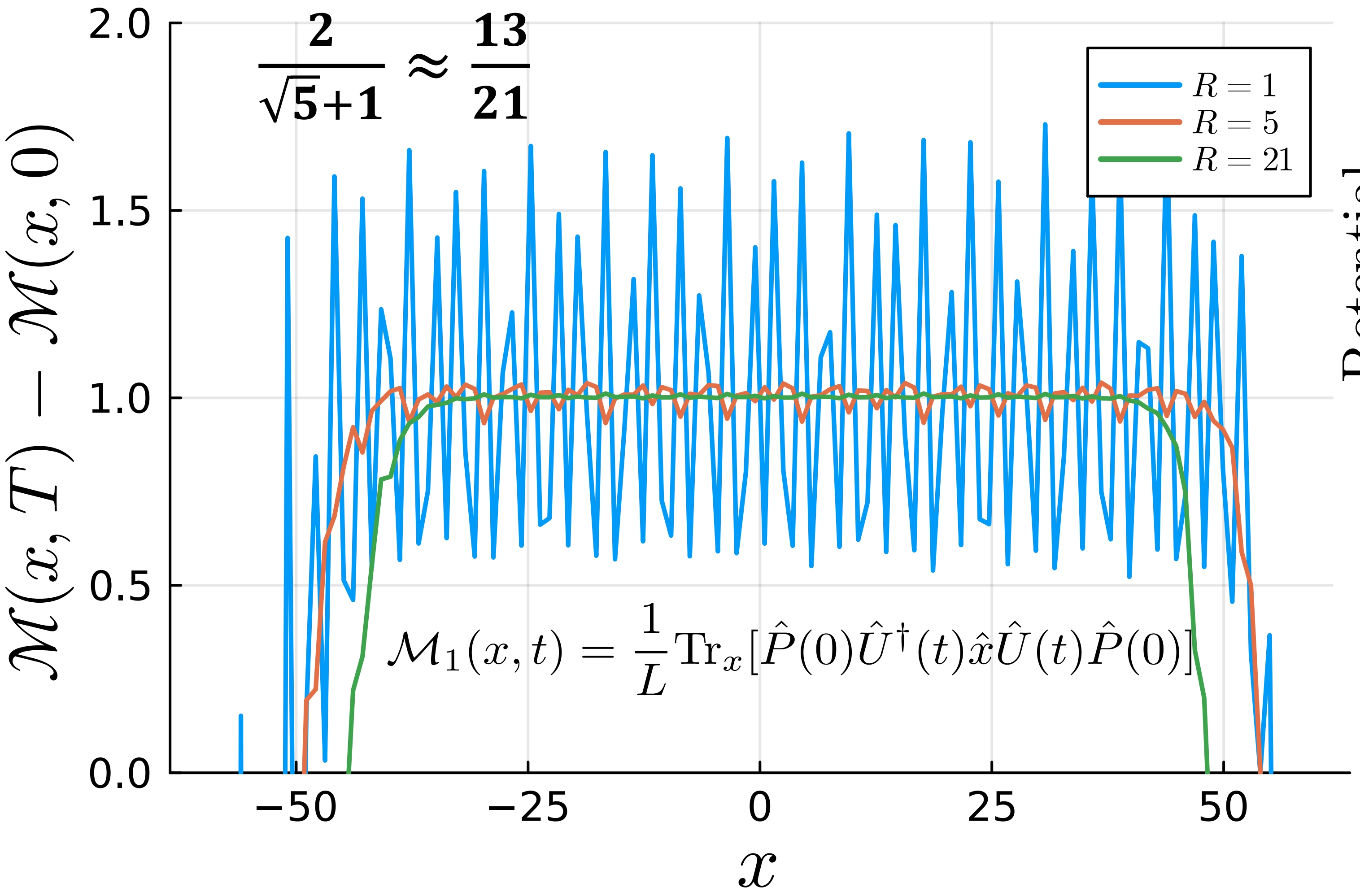
Results of $\beta = \frac{3}{5}$: Periodic Case

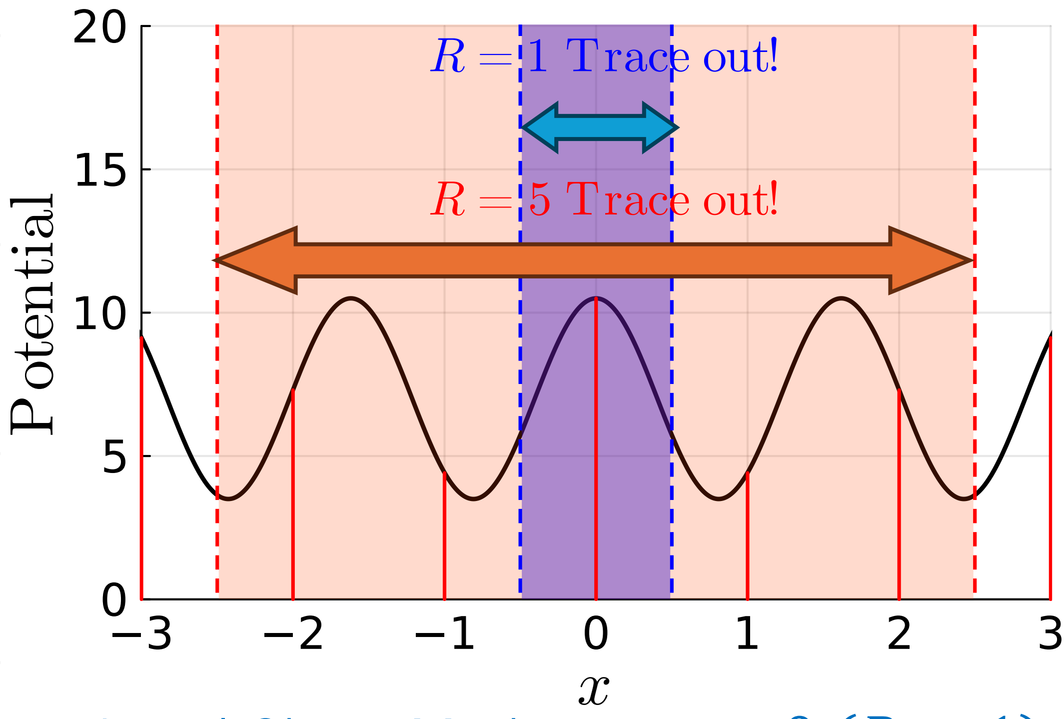


Results of $\beta = \frac{2}{\sqrt{5}+1}$: Non-periodic case



Results of $\beta = \frac{2}{\sqrt{5}+1}$: Non-periodic case

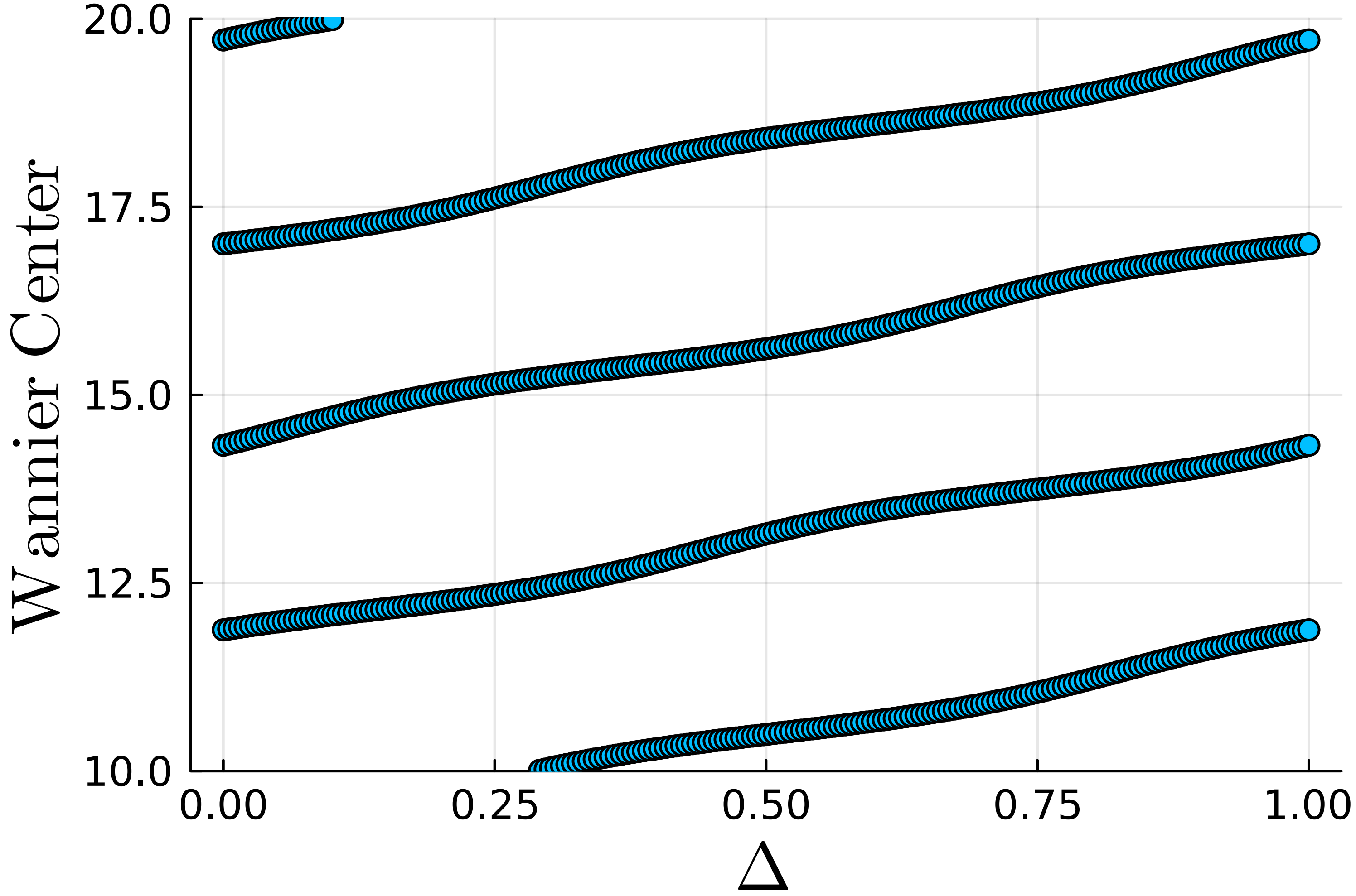




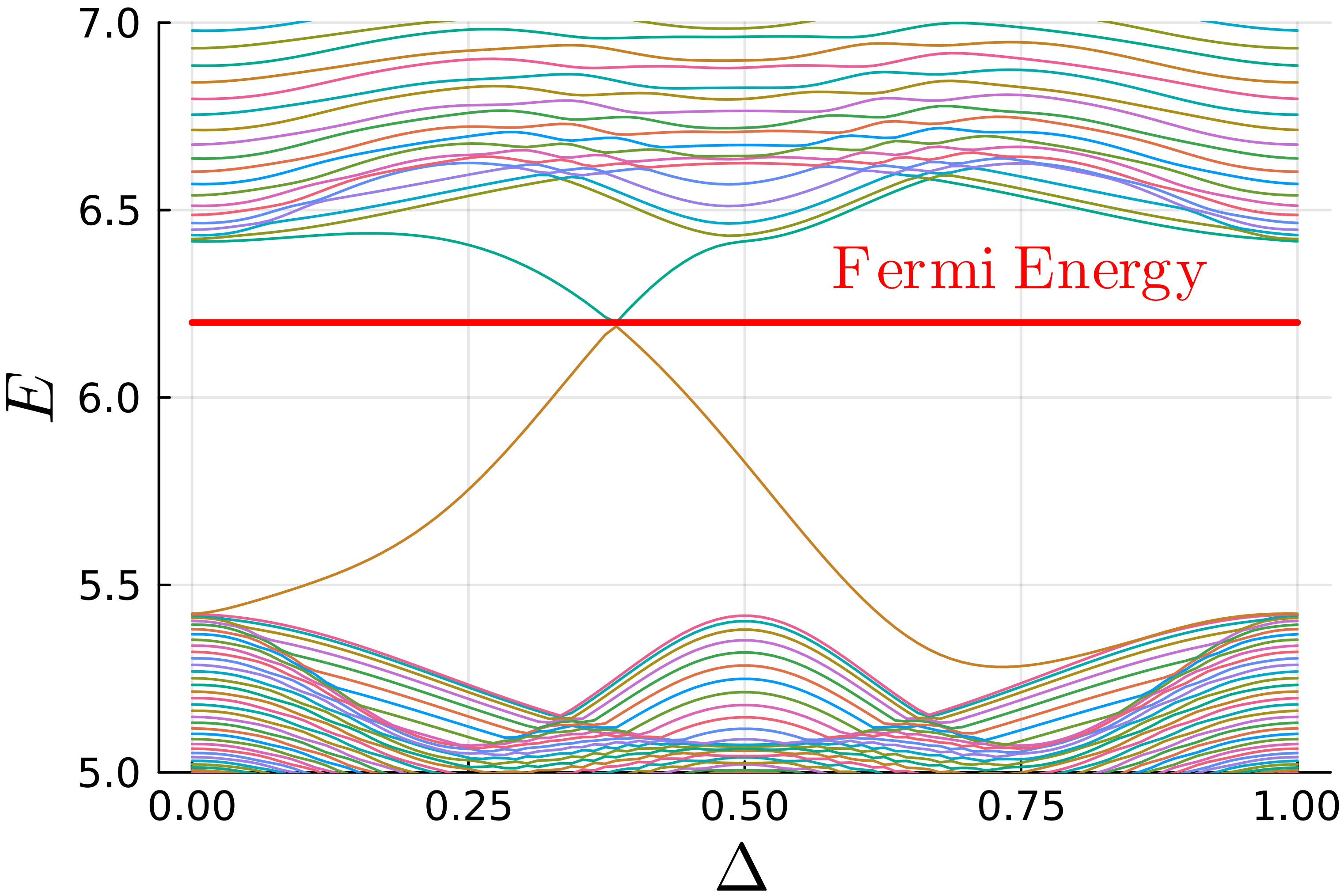
Local Chern Marker at $x = 0$ $(R = 1)$

Local Chern Marker at $x = 0$ $(R = 5)$

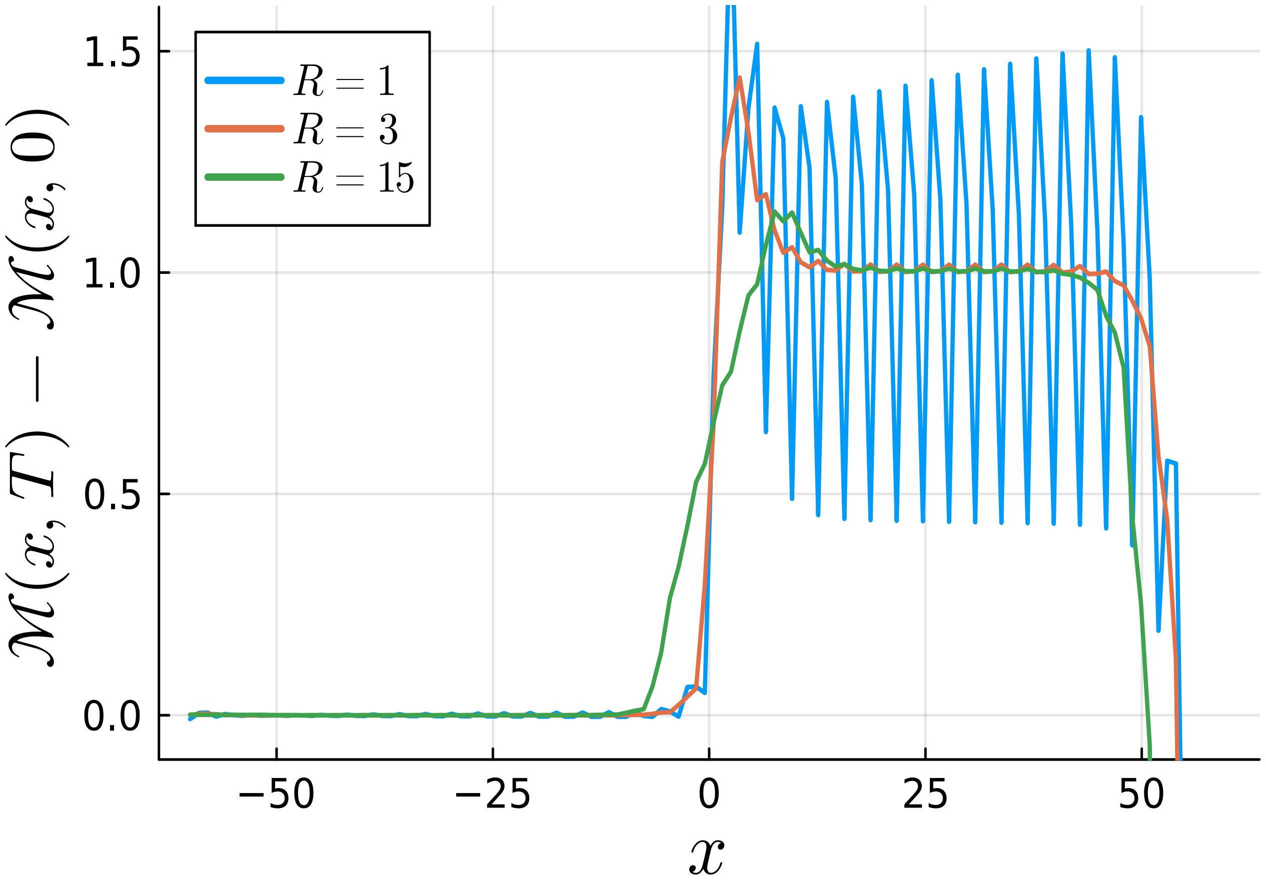
Results of $\beta = \frac{2}{\sqrt{5}+1}$: Non-periodic case

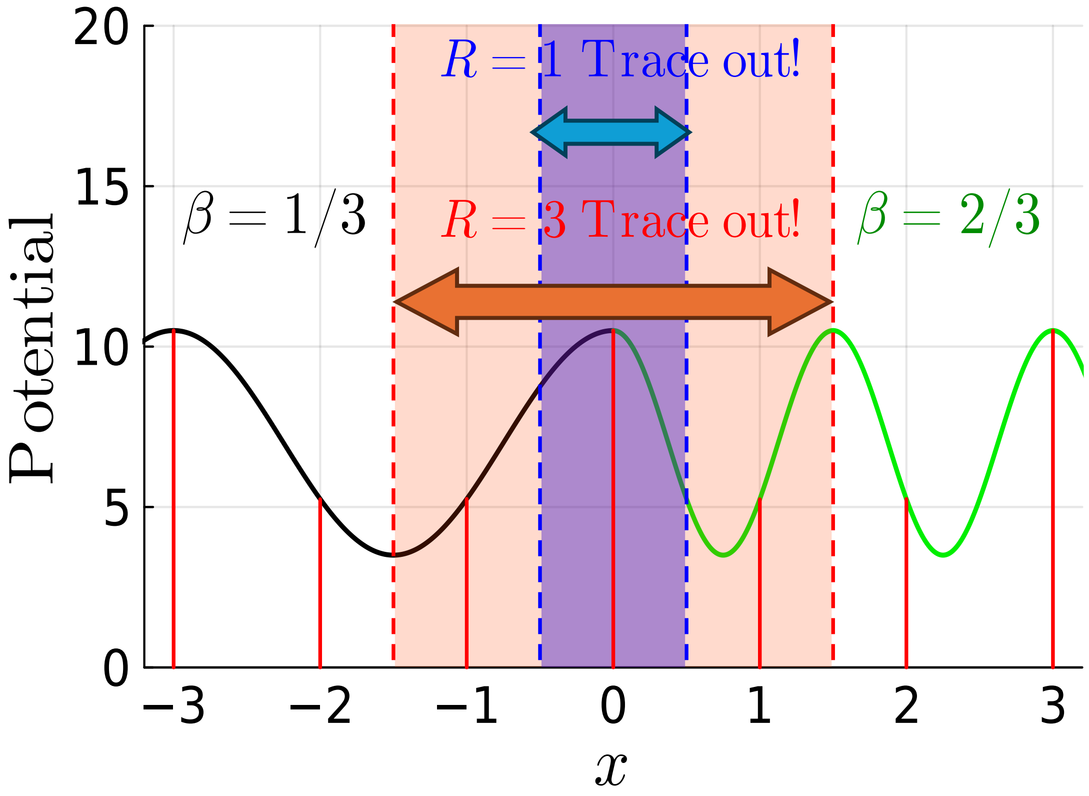


Results of $\beta = \frac{1}{3}, \frac{2}{3}$: Domain Wall Case

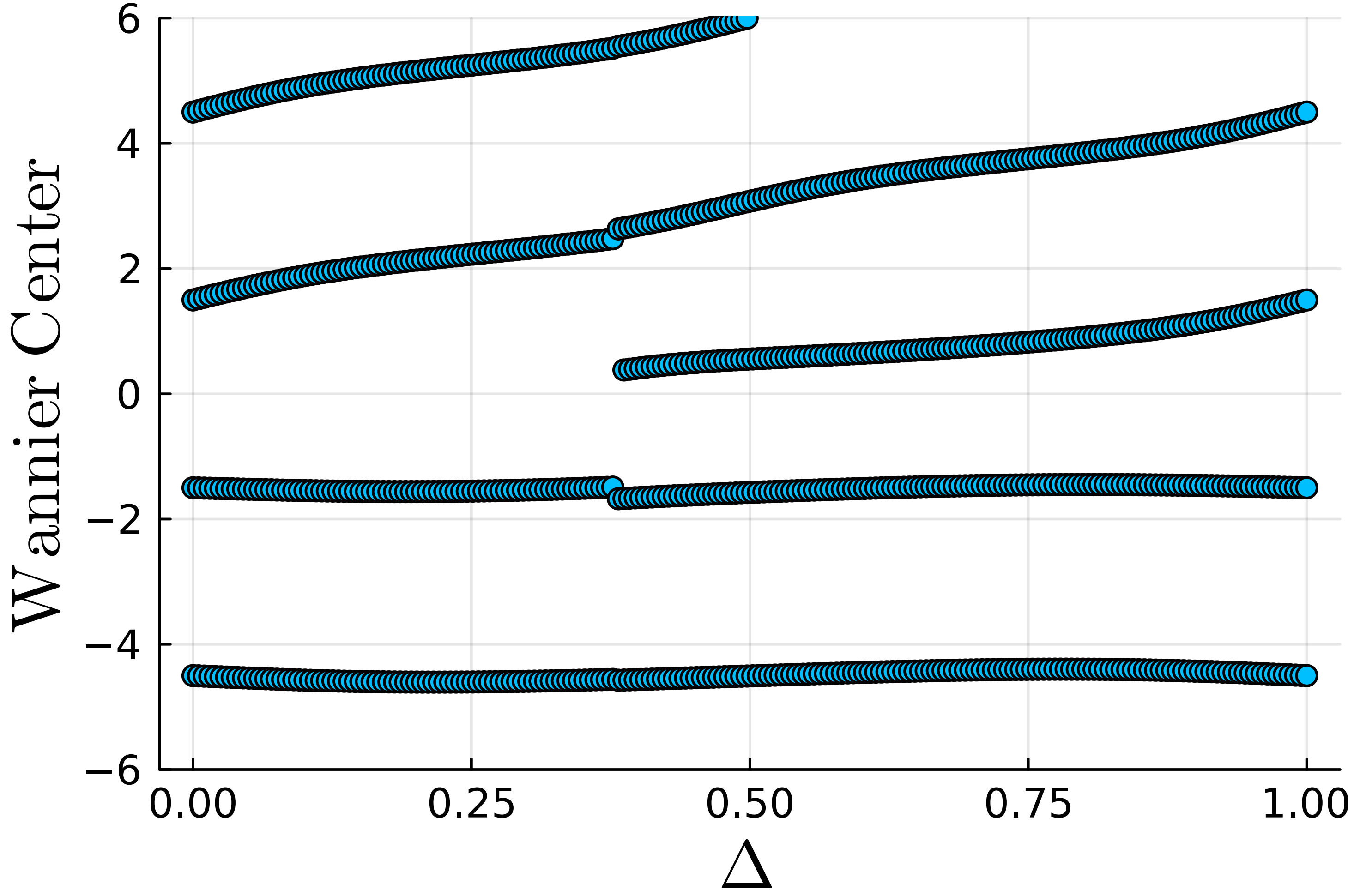


Results of $\beta = \frac{1}{3}, \frac{2}{3}$: Domain Wall Case





Results of $\beta = \frac{1}{3}, \frac{2}{3}$: Domain Wall Case



Acknowledgment

Thank you!

ありがとうございました！